Lecture 35

$\mathsf{ZPP} = \mathsf{RP} \cap \mathsf{coRP}$, PIT

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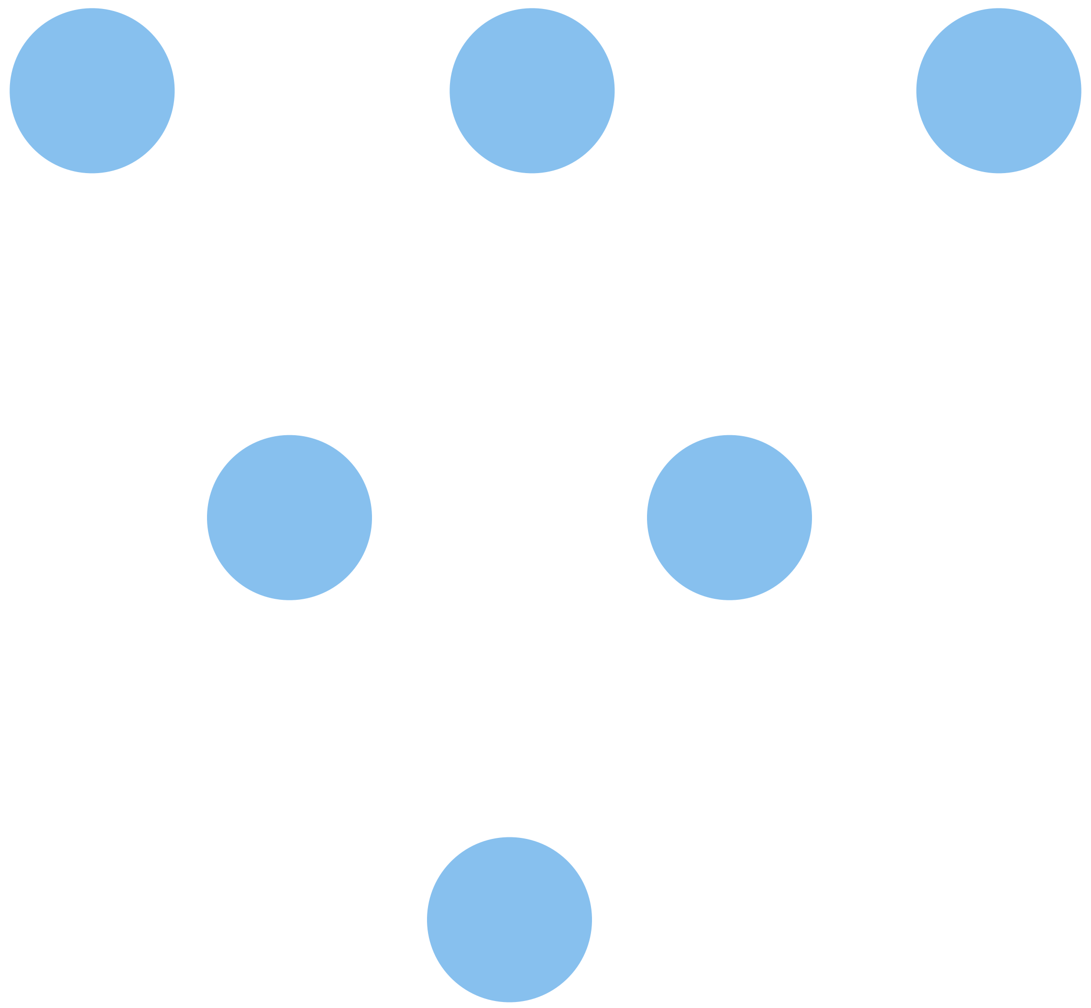

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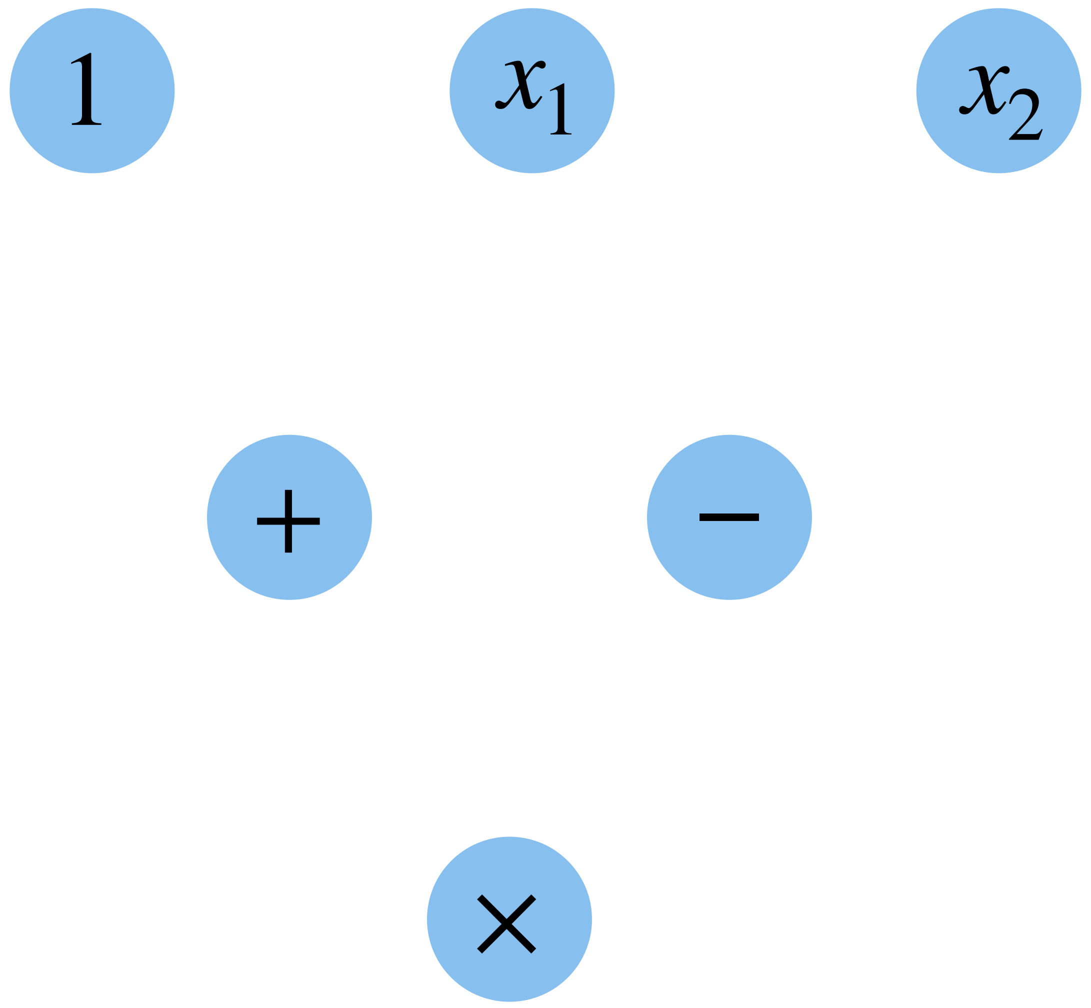

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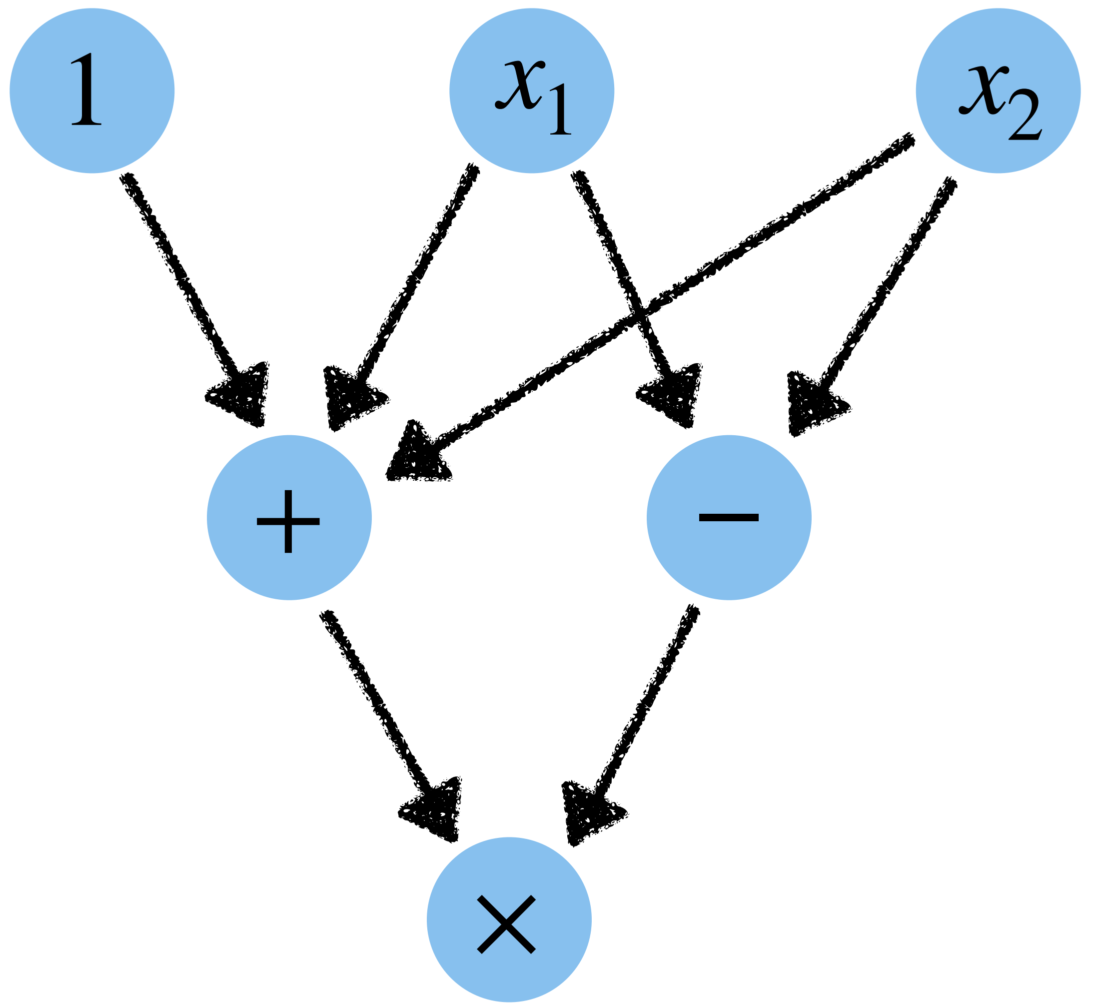

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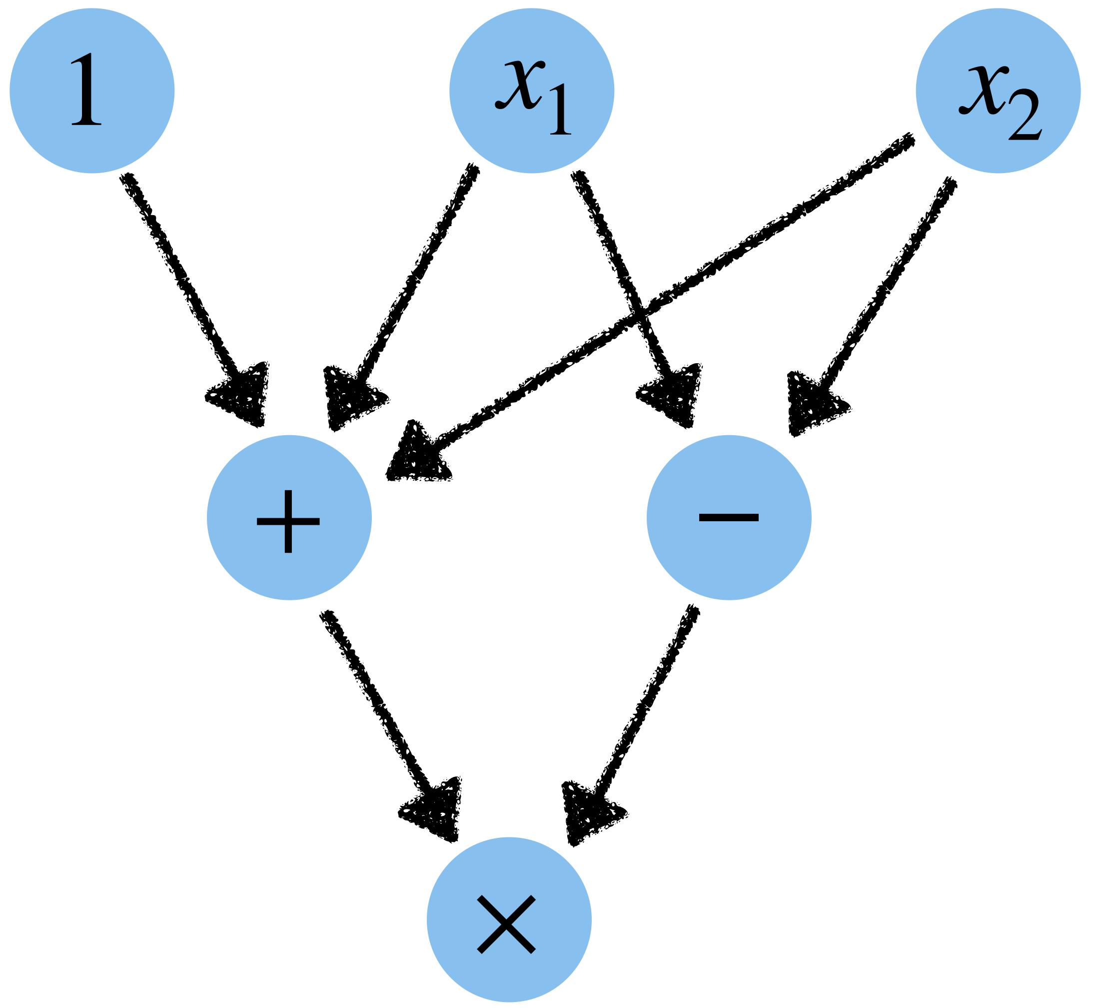

$$= \quad (x_1 + x_2 + 1)(x_1 - x_2)$$

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Flaw: $O(n)$ size algebraic circuits can compute polynomials with 2^n many monomials, e.g., $\Pi_{i\in[n]}(1+x_i)$.

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$$\mathbf{Pr}[p(a_1, a_2, \ldots, a_m) \neq 0] \geq 1 - \frac{d}{|S|}$$

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Flaw: y can be as large as $(10.2^m)^{2^m}$.

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How can we improve the probability to a constant?

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How can we improve the probability to a constant?

Repeat B $O(m)$ times and accept iff $y = 0$ on all iterations.

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4) Repeat 2), 3) $40m/9$ times.

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Algorithm D for *ZEROP* for a circuit C of size m that takes n input:

1) Choose $x_1, x_2, \ldots, x_n$ randomly from $[1,2,\ldots\ldots,10.2^m]$.

2) Choose k randomly from $[1,2^{2m}]$.

3) Evaluate $C(x_1, x_2, \ldots\ldots, x_n)$ mod $k = y$.

4) Repeat 2), 3) $40m/9$ times.

5) Accept iff $y = 0$ on all iterations.

Correctness Analysis:

If $C \in$ *ZEROP*, then A accepts with probability 1.

If $C \notin$ *ZEROP*, then A accepts with probability $\leq \left(1 - \frac{9}{40m}\right)$

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